THE ISOMETRY GROUP OF $L^p(\mu, \mathrm{X})$ IS SOT-CONTRACTIBLE

JARNO TALPONEN

Abstract. We will show that if (Ω, Σ, μ) is an atomless positive measure space, X is a Banach space and $1 \leq p < \infty$, then the group of isometric automorphisms on the Bochner space $L^p(\mu, \mathrm{X})$ is contractible in the strong operator topology. We do not require Σ or X above to be separable.

1. Introduction

This article deals with the topological structure of isometry groups of Banach spaces. Recall that an isometry group $\mathcal{G}_{\mathrm{X}}$ of a Banach space X consists of linear isometric isomorphisms $T \colon \mathrm{X} \to \mathrm{X}$.

The connectedness of groups of linear automorphisms with respect to the norm topology is a classical topic by now, see e.g. [4, 13]. For example, Kuiper proved already in 1965 that the linear automorphism group GL_{ℓ^2} and the isometry group $\mathcal{G}_{\ell^2}$ of ℓ^2 are operator norm contractible. On the other hand, spaces $\mathcal{G}_{\ell^p}$ and $\mathcal{G}_{L^p(\mu)}$ are discrete in the operator norm topology for $1 \leq p \leq \infty,\ p \neq 2$, ([8, p.112], [1, p.57]) and a fortiori not contractible. Our main result result shows that the situation is surprisingly different when observing $\mathcal{G}_{L^p(\mu)}$ in the strong operator topology:

Theorem 1.1. *Let* X *be a Banach space,* μ *an atomless positive measure and* $1 \leq p < \infty$. *Then the isometry group* $\mathcal{G}_{L^p(\mu,\mathrm{X})}$ *of the Bochner space* $L^p(\mu, \mathrm{X})$ *endowed with the strong operator topology is contractible. Here* X *and* $L^p(\mu, \mathrm{X})$ *can be regarded as real or complex spaces.*

This result involves the isometric structure and more precisely the L^p-structure of Bochner spaces. For a fascinating treatment of the latter topic see [2, 9] which are implicitly applied in this work. Our study is also closely related to [10], [11] and [12].

Recall that for Banach spaces X and Y a projection $P \colon \mathrm{X} \oplus \mathrm{Y} \to \mathrm{Y}$ is called an L^p-projection for a given $p \in [1, \infty)$ if

$$||(x, y)||^p_{\mathrm{X}\oplus\mathrm{Y}} = ||x||^p_{\mathrm{X}} + ||y||^p_{\mathrm{Y}} \quad \text{for each } x \in \mathrm{X},\ y \in \mathrm{Y}.$$

Such projections commute and in fact the L^p-structure $\mathbb{P}_p(\mathrm{X})$ of X, i.e. the set of all L^p-projections on X can be regarded as a complete Boolean algebra (see [2]).

In order to put our result in the right context, let us mention that our route to Theorem 1.1 is via analyzing the L^p-structure of $L^p(\mu, \mathrm{X})$. This structure depends on μ and the L^p-structure of X, as one might expect. If X is separable, then one can write $\mathbb{P}_p(L^p(\mu, \mathrm{X})) = \Sigma/\mu \otimes \mathbb{P}_p(\mathrm{X})$ in a suitable sense, see [12]. It might first seem, bearing the L^p-structures in mind, that an effective means to

Date: April 28, 2008.
1991 *Mathematics Subject Classification.* Primary 46B04; Secondary 46B25, 46E40.

analyze the connectedness properties of $\mathcal{G}_{L^p(\mu,\mathrm{X})}$ would be applying suitably normalized L^p-decompositions of Bochner spaces. For example the p-integral module representation (see [2]) appears to be a natural tool for such an approach.

In fact, the L^p-structure of $L^p(\mu,\mathrm{X})$ is simple to represent if X is a separable Banach space having only trivial L^p-structure and $1\leq p<\infty,\ p\neq 2$. Then the isometries $T\in\mathcal{G}_{L^p(\mu,\mathrm{X})}$ can be represented, in a suitable sense, as

$$Tf(t)=\sigma_t\left(\frac{d(\mu\circ\phi^{-1})}{d\mu}(t)\right)^{\frac{1}{p}} f\phi^{-1}(t)\ \text{for}\ f\in L^p(\Omega,\Sigma,\mu,\mathrm{X}), \tag{1.1}$$

where $\sigma\colon\Omega\to\mathcal{G}_{\mathrm{X}};\ t\mapsto\sigma_t$ is strongly measurable and $\phi\colon\Sigma/\mu\leftrightarrow\Sigma/\mu$ is a Boolean isomorphism (see [9] and also [10]). Moreover, there is in general a close connection between the L^p-structure of a given Lebesgue-Bochner space and the corresponding representations of type (1.1).

However, there exists an obstruction, that has to be dealt with. Namely, in the setting of Theorem 1.1 the space X may have a rich L^p-structure and be non-separable, so that typically the L^p-structure of $L^p(\mu,\mathrm{X})$ is very complicated and (1.1) fails. Another substantial difficulty is that if X has a trivial L^p-structure but is *non-separable*, then the L^p-structure of $L^p(\mu,\mathrm{X})$ is not known explicitly, nor is it known whether representation (1.1) holds. Thus we note that even though the group $\mathcal{G}_{L^p(\mu,\mathrm{X})}$ has not been classified in the sense of L^p-structures, we are unexpectedly able to extract enough information of $\mathbb{P}_p(L^p(\mu,\mathrm{X}))$ in order to establish a very strong connectedness condition in Theorem 1.1.

The way around the described problems is to employ a suitable isometric representation for $L^p(\mu,\mathrm{X})$, which rises from measure-theoretic observations.

To introduce the notations applied in this paper, X, Y and Y stand for *real* Banach spaces. The closed unit ball and the unit sphere of X are denoted by $\mathbf{B}_{\mathrm{X}}$ and $\mathbf{S}_{\mathrm{X}}$, respectively. We refer to [5] and [6] for necessary background information in measure theory and isometric theory of classical Banach spaces. It is also useful to get acquainted with the machinery appearing in [9] regarding Bochner spaces.

If $\mathcal{F}\subset\mathcal{P}(\Omega)$ and $\Sigma_0\subset\Sigma$ is a σ-subring such that $\mathcal{F}\subset\Sigma_0$ and for all $A\in\Sigma_0$ there is a set $\{B_n|n\in\mathbb{N}\}$, where each B_n is a countable intersection of suitable elements of $\mathcal{F}$, and $\bigcup_n B_n=A$, then we say that $\mathcal{F}$ *σ-generates* Σ_0. Recall that the strong operator topology (SOT) on $L(\mathrm{X})$ is the topology inherited from X^{X} endowed with the product topology. Recall that each $T\in L(\mathrm{X})$ has a SOT-neighbourhood basis consisting of sets of the following type:

$$\{S\in L(\mathrm{X}):||(S-T)x_1||,||(S-T)x_2||,\ldots,||(S-T)x_n||<\epsilon\}, \tag{1.2}$$

where $x_1,\ldots,x_n\in\mathrm{X},\ n\in\mathbb{N}$ and $\epsilon>0$.

Let us make some preparations towards the proof of Theorem 1.1 by introducing some auxiliary notions. In what follows (Ω,Σ,μ) is a positive measure space, where Σ is a σ-ring. An arbitrary measure space may be unconveniently rich for our purposes, that is, for studying the bands of $L^p(\Omega,\Sigma,\mu)$ for $p\in[1,\infty)$. Therefore we wish to extract exactly the information which is 'recoqnized' by the L^p-structure. If Σ is as above, then we will define

$$\Sigma^L=\{f^{-1}(\mathbb{R}\setminus\{0\})|f\in L^p(\Omega,\Sigma,\mu)\}/\mu.$$

Above $\{f^{-1}(\mathbb{R}\setminus\{0\})|f\in L^p(\Omega,\Sigma,\mu)\}$ is a σ-subring of Σ and Σ^L is the quotient σ-ring formed by identifying μ-null sets with $\emptyset$. By slight abuse of notation we denote the corresponding measure ring by (Σ^L,μ) and we consider the elements $A\in\Sigma^L$ as

contained in Ω in the μ-a.e. sense. Hence, for $A, B \in \Sigma^L$, we will write $A \bigcup B \in \Sigma^L$ instead of $A \bigvee B \in \Sigma^L$, and so on. Note that Σ^L does not depend on the value of p, as long as $p < \infty$, since $\{f^{-1}(\mathbb{R} \setminus \{0\}) | f \in L^p(\Omega, \Sigma, \mu)\} \subset \Sigma$ is just the σ-subring of σ-finite sets. The motivation of this concept becomes clear in the subsequent results. Given $p \in [1, \infty)$, a set I and Banach spaces X_i for $i \in I$, we denote by $\bigoplus_{i \in I}^p \mathrm{X}_i$ the L^p-sum of the spaces X_i. That is, $(x_i)_{i \in I} \in \prod_{i \in I} \mathrm{X}_i$ satisfies $(x_i)_{i \in I} \in \bigoplus_{i \in I}^p \mathrm{X}_i$ if and only if $||(x_i)_{i \in I}||^p \doteq \sum_{i \in I} ||x_i||_{\mathrm{X}_i}^p < \infty$. The space $\bigoplus_{i \in I}^p \mathrm{X}_i$ is endowed with the complete norm $|| \cdot ||$. We will denote by $P_j \colon \bigoplus_{i \in I}^p \mathrm{X}_i \to \mathrm{X}_j$ the L^p-projection onto X_j for each $j \in I$, where X_i is regarded as a subspace of $\bigoplus_{i \in I}^p \mathrm{X}_i$ in the natural way. Hence each $x \in \mathrm{X}_j$ is thought of as an element of $\bigoplus_{i \in I}^p \mathrm{X}_i$ as $P_j x = x$.

The Lebesgue measure on $[0, 1]$ is denoted by m and if κ is a non-zero cardinal, then we denote the product measure on $[0, 1]^\kappa$ by $m^\kappa \colon \Sigma_{[0,1]^\kappa} \to \mathbb{R}$, where $\Sigma_{[0,1]^\kappa}$ is the corresponding product σ-algebra.

The *orbit* of $x \in \mathbf{S}_\mathrm{X}$ is $\mathcal{G}_\mathrm{X}(x) \doteq \{T(x) | T \in \mathcal{G}_\mathrm{X}\}$.

2. Results

We will require the following auxiliary results.

Lemma 2.1. *For given $f \in L^p(m, \mathrm{X})$ and $t_0 \in [0, 1)$ it holds that*

$$\lim_{\substack{t \to t_0 \\ t \in [0,1]}} \int_{t_0}^1 ||f(h_t(s)) - f(h_{t_0}(s))||^p \, \mathrm{d}s = 0,$$

where $h_t \colon [t_0, 1] \to [t, 1]$; $h_t(s) = \left(\frac{t - t_0}{1 - t_0} + \frac{1 - t}{1 - t_0} s\right)$ for $t, s \in [0, 1]$.

Proof. The claim reduces to the analogous scalar-valued statement by approximating f with simple functions. This in turn can be obtained by a straightforward modification of the proof of the classical fact that $\lim_{h \to 0} \int_\mathbb{R} |g(s+h) - g(s)| \, \mathrm{d}s = 0$ for $g \in L^p(\mathbb{R})$, $p < \infty$, which exploits Lusin's theorem. $\square$

Lemma 2.2. *Let X be a Banach space, (Ω, Σ, μ) be an atomless positive measure space and $p \in [1, \infty)$. Then there exists a set I such that $L^p(\mu, \mathrm{X})$ is isometrically isomorphic to $\bigoplus_{i \in I}^p L^p(m, L^p(m^{\lambda_i}, \mathrm{X}))$, where λ_i are non-zero cardinals for $i \in I$.*

Proof. The argument is closely related to classical matters discussed in [6] and [12]. Recall Lamperti's classical result [7] that the supports of $f, g \in L^1(\mu)$ are essentially disjoint if and only if $||f + g|| + ||f - g|| = 2(||f|| + ||g||)$. The crucial conclusion of this result is that the disjointness of two vectors $f, g \in L^p(\nu)$ can be detected by studying the above norms and hence each linear isometry $\psi \colon L^1(\nu) \to L^1(\upsilon)$, where ν and υ are positive measures, preserves disjointness and bands. This also leads to the fact that a projection P on $L^1(\mu)$ is L^1-projection with a separable range if and only if there is $A \in \Sigma^L$ such that the image of P is $\{f \in L^1(\mu) | \mathrm{supp}(f) \subset A \ \mu\text{−a.e.}\}$.

Since each $f \in L^1(\mu)$ is σ-finitely supported, it follows that the measure ring (Σ^L, μ) is σ-generated by μ-finite sets. Recall that the Boolean algebra of L^1-projections on $L^1(\mu)$ is complete (see [2, Prop.1.6]). Thus, by recalling Lamperti's result we obtain that $\{A \setminus \bigcup \mathcal{F} | A \in \Sigma^L\}$ defines an ideal of Σ^L for any $\mathcal{F} \subset \Sigma^L$. Hence Hausdorff's maximum principle yields a maximal family $\{V_j\}_{j \in J} \subset \Sigma^L$ of pairwise μ-essentially disjoint μ-finite sets. Note that $\bigoplus_{j \in J}^1 L^1(\mu|_{V_j})$ is isometric to $L^1(\mu)$.

By using the μ-finiteness of the sets V_j we obtain that for each $j \in J$ there is a countable set A_j and cardinals λ_k for $k \in A_j$ such that the subspace $\{f \in L^1(\mu)|\mathrm{supp}(f) \subset V_j\ \mu - \text{a.e.}\}$ is isometric to $\bigoplus_{k\in A_j}^1 L^1(m^{\lambda_k})$ (see [6, p.127]). Since V_j are essentially disjoint, the sets A_j can be chosen to be pairwise disjoint.

Put $I = \bigcup_{j\in J} A_j$. Observe that there exists an isometric isomorphims $\phi\colon \bigoplus_{i\in I}^1 L^1(m^{\lambda_i}) \to L^1(\mu)$. According to Lamperti's result the map ϕ preserves bands. Recall that if $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are positive σ-finite measure spaces such that $L^1(\mu_1)$ and $L^1(\mu_2)$ are isometric, then there is a Boolean isomorphism $\tau\colon \Sigma_1/\mu_1 \leftrightarrow \Sigma_2/\mu_2$ see e.g. [12, p.477]. It follows from the selection of $\{V_j\}_{j\in J}$ that there exist ideals $B_i \subset \Sigma^L$ for $i \in I$ such that

(a) Σ^L is σ-generated by $\bigcup_{i\in I} B_i$,
(b) B_i is Boolean isomorphic to $\Sigma_{[0,1]^{\lambda_i}}/m^{\lambda_i}$ for $i \in I$
(c) the ideals B_i are pairwise essentially disjoint.

Indeed, each ideal B_i is determined by the isometric embedding

$$\phi|_{L^1(m^{\lambda_i})}\colon L^1(m^{\lambda_i}) \hookrightarrow L^1(\mu) \quad \text{for } i \in A_j,$$

namely, B_i is the essential support of the functions in the image of $\phi|_{L^1(m^{\lambda_i})}$.

Let $p \in [1, \infty)$. For each $i \in I$ there is a Boolean isomorphism from $\Sigma_{[0,1]^{\lambda_i}}/m^{\lambda_i}$ onto the corresponding ideal B_i of Σ^L. Clearly

$$\{f \in L^p(\mu, \mathrm{X})|\mathrm{supp}(f) \subset B_i\ \mu - \text{a.e.}\} \subset L^p(\mu, \mathrm{X})$$

is a closed subspace isometric to $L^p(\mu|_{B_i}, \mathrm{X})$. Thus there exists an isometric isomorphism $U_i\colon L^p(m^{\lambda_i}, \mathrm{X}) \to \{f \in L^p(\mu, \mathrm{X})|\mathrm{supp}(f) \subset B_i\ \mu - \text{a.e.}\}$ for $i \in I$, see e.g. [12, p.476].

Define an isometric isomorphism $U\colon \bigoplus_{i\in I}^p L^p(m^{\lambda_i}, \mathrm{X}) \to L^p(\mu, \mathrm{X})$ by $(f_i)_{i\in I} \mapsto \sum_{i\in I} U_i(f_i)$. Indeed, this mapping is an isometry since the ideals B_i are disjoint. Moreover, as Σ^L is σ-generated by the ideals B_i it is easy to see by analyzing simple functions of $L^p(\mu, \mathrm{X})$ that U is onto.

The following final step finishes the proof. We claim that if κ is a non-zero cardinal, then $L^p(m^\kappa, \mathrm{X})$ is isometrically isomorphic to $L^p(m, L^p(m^\kappa, \mathrm{X}))$. Indeed, recall that each $m \otimes m^\kappa$-measurable set $A \subset [0,1] \times [0,1]^\kappa$ can be approximated in measure by countable unions of measurable rectangles (see e.g. [5, p.145]). For the scalar-valued case, see [6, p.127] and [12, p.478-479]. Hence the obvious identification of the spaces $L^p(m \otimes m^\kappa, \mathrm{X})$ and $L^p(m, L^p(m^\kappa, \mathrm{X}))$ can be obtained, since each simple function in $L^p(m \otimes m^\kappa, \mathrm{X})$ can be approximated by a sequence of simple functions of $L^p(m, L^p(m^\kappa, \mathrm{X}))$ and vice versa. □

Now we are ready to prove our main result.

Proof of Theorem. 1.1. Since μ is atomless we may apply Lemma 2.2 to $L^p(\mu, \mathrm{X})$ for $1 \leq p < \infty$. Thus we may write $L^p(\mu, \mathrm{X}) = \bigoplus_{i\in I}^p L^p(m, L^p(m^{\kappa_i}, \mathrm{X}))$ isometrically, where κ_i are non-zero cardinals for $i \in I$. In what follows we will denote $\mathcal{M} \doteq \bigoplus_{i\in I}^p L^p(m, L^p(m^{\kappa_i}, \mathrm{X}))$ for the sake of brevity. Hence it suffices to show that $\mathcal{G}_{\mathcal{M}}$ is SOT-contractible.

For each $i \in I$ we define mappings

$$\alpha_i, \beta_i, \gamma_i\colon [0,1] \times L^p(m, L^p(m^{\kappa_i}, \mathrm{X})) \to L^p(m, L^p(m^{\kappa_i}, \mathrm{X}))$$

by $\alpha_i(t, f_i) = \chi_{[0,t]} f_i$,

$$\beta_i(t, f_i)(s) = (1-t)^{-\frac{1}{p}} f_i(t + (1-t)s) \text{ and } \gamma_i(t, f_i) \circ \beta_i(t, f_i) = \chi_{[t,1]} f_i,$$

where $f_i \in L^p(m, L^p(m^{\kappa_i}, \mathrm{X}))$ and $s \in [0,1]$. Above we apply the convention $0^{-\frac{1}{p}} = 0$. Clearly $\alpha_i(t, \cdot), \beta_i(t, \cdot)$ and $\gamma_i(t, \cdot)$ are contractive linear operators for $i \in I,\ t \in [0,1]$. Note that $\alpha_i(\cdot, f_i)$ are continuous on $[0,1]$ and according to Lemma 2.1 the same is true for $\beta_i(\cdot, f_i)$ and $\gamma_i(\cdot, f_i)$.

The required homotopy $h\colon [0,1] \times \mathcal{G}_{\mathcal{M}} \to \mathcal{G}_{\mathcal{M}}$ is given by

$$h(t,T)\left(\textstyle\sum_{i\in I} f_i\right) = \textstyle\sum_{i\in I} \alpha_i(t, f_i) + \textstyle\sum_{i\in I} \gamma_i\left(t, P_i T\left(\textstyle\sum_{i\in I} \beta_i(t, f_i)\right)\right)$$

for $t \in [0,1],\ T \in \mathcal{G}_{\mathcal{M}}$ and $\sum_{i\in I} f_i \in \mathcal{M}$. Indeed, it is straightforward to check that $h(t,T) \in \mathcal{G}_{\mathcal{M}}$ for each $t \in [0,1]$ and $T \in \mathcal{G}_{\mathcal{M}}$. Moreover, $h(0,T) = T$ and $h(1,T) = \mathrm{id}$ for each $T \in \mathcal{G}_{\mathcal{M}}$. Our next aim is to justify that h is indeed a suitable homotopy with respect to the strong operator topology.

Let $t_0 \in [0,1]$, $T \in \mathcal{G}_{\mathcal{M}}$ and let $V \subset \mathcal{G}_{\mathcal{M}}$ be a SOT-open neighbourhood of $h(t_0, T)$. In order to justify the $|\cdot| \times \mathrm{SOT} - \mathrm{SOT}$ continuity of h at (t_0, T), we must find an open set $W \subset ([0,1], |\cdot|) \times (\mathcal{G}_{\mathcal{M}}, \mathrm{SOT})$ such that $(t_0, T) \in W$ and $h(W) \subset V$. Fix $\epsilon > 0$ and $f = \sum_{i\in I} f_i, g = \sum_{i\in I} g_i \in \mathbf{S}_{\mathcal{M}}$ such that $||g - h(t_0,T)(f)|| < \epsilon$.

Recall that the point $h(t_0, T)$ has a SOT$-$neighbourhood basis of type (1.2), and the elementary topological fact that open sets are preserved in finite intersections. Since ϵ, f and g were arbitrary, it suffices to show that there are open sets $\Delta \subset ([0,1], |\cdot|)$ and $U \subset (\mathcal{G}_{\mathcal{M}}, \mathrm{SOT})$ such that $t_0 \in \Delta,\ T \in U$ and

$$h(\Delta \times U) \subset \{R \in \mathcal{G}_{\mathcal{M}} :\ ||g - Rf|| < \epsilon\}. \tag{2.1}$$

Denote $\delta = \epsilon - ||g - h(t_0,T)f|| > 0$. There exists a finite set $J \subset I$ such that $||\sum_{i\in I\setminus J} f_i|| < \frac{\delta}{4}$. Put $n = |J|$. Fix $j \in J$. Similarly as above, let $f_j = P_j f$. Since $\beta_j(\cdot, f_j)$ is continuous we can find an open interval $\Delta_j \subset [0,1]$ containing t_0 such that $||\beta_j(t, f_j) - \beta_j(t_0, f_j)|| < \frac{\delta}{4n}$ for $t \in \Delta_j$. Define an SOT$-$open neighbourhood U_j of T by

$$U_j = \{S \in \mathcal{G}_{\mathcal{M}} :\ ||(S-T)\beta_j(t_0, f_j)|| < \frac{\delta}{4n}\}.$$

Finally, by recalling the definitions of h, Δ_j and U_j we obtain that

$$\begin{aligned} ||h(t,S)f_j - h(t_0,T)f_j|| &\le ||h(t,S)f_j - h(t_0,S)f_j|| + ||h(t_0,S)f_j - h(t_0,T)f_j|| \\ &< \tfrac{\delta}{4n} + ||(S-T)\beta_j(t_0,f_j)|| < \tfrac{\delta}{2n} \end{aligned}$$

for $t \in \Delta_j$ and $S \in U_j$, where $j \in J$. Put $\Delta = \bigcap_{j\in J} \Delta_j$ and $U = \bigcap_{j\in J} U_j$. We get

$$\begin{aligned} &||h(t,S)f - h(t_0,T)f|| \\ &\le \left|\left|(h(t,S) - h(t_0,T)) \textstyle\sum_{i\in I\setminus J} f_i\right|\right| + \textstyle\sum_{j\in J} ||h(t,S)f_j - h(t_0,T)f_j|| \\ &< ||h(t,S) - h(t_0,T)|| \ \cdot\ \left|\left|\textstyle\sum_{I\setminus J} f_i\right|\right| + n\tfrac{\delta}{2n} < \delta \end{aligned}$$

for $(t,S) \in \Delta \times U$. This means that $||g - h(t,S)f|| < ||g - h(t_0,T)f|| + \delta = \epsilon$ for $(t,S) \in \Delta \times U$. Consequently $h(\Delta \times U)$ satisfies (2.1) and the proof is complete. □

Recall that $\mathbf{S}_{L^p(m)}$, $1 \le p < \infty$, consists of (exactly) 2 orbits, namely $\mathcal{G}_{L^p}(\chi_{[0,1]})$ and $\mathcal{G}_{L^p}(2^{\frac{1}{p}}\chi_{[0,\frac{1}{2}]})$, both of which are dense in $\mathbf{S}_{L^p(m)}$ (see e.g. [14]).

Corollary 2.3. *Both the orbits $\mathcal{G}_{L^p}(\chi_{[0,1]}), \mathcal{G}_{L^p}(2^{\frac{1}{p}}\chi_{[0,\frac{1}{2}]})$, $1 \le p < \infty$, are path-connected.*

Proof. Fix 2 points $f, g \in \mathbf{S}_{\mathrm{X}}$ both coming from one of the above orbits. Then there exists $T \in \mathcal{G}_{L^p}$ such that $T(f) = g$. According to Theorem 1.1 there is a homotopy $h\colon [0,1] \times \mathcal{G}_{L^p} \to \mathcal{G}_{L^p}$ such that $h(0,T) = T$ and $h(1,T) = \mathrm{id}$. Clearly $h([0,1] \times \mathcal{G}_{L^p})(\cdot)$ preserves orbits. Note that $t \mapsto h(t,T)f$ defines a continuous path connecting g and f in $\mathcal{G}_{L^p}(f) \subset \mathbf{S}_{L^p}$. $\square$

The assumptions of μ being atomless or $p < \infty$ cannot be removed in Theorem 1.1 even in the scalar-valued case. If $p \in [1, \infty)$, $p \neq 2$, and μ has some atoms, then by applying the scalar-valued analogue of representatation (1.1) (see [6]) it can be verified that $\mathcal{G}_{L^p(\mu)}$ is not connected in the weak operator topology. However, if ν is the counting measure on $\mathbb{N}$, then $L^p(\nu, L^p(m)) = L^p(m)$ isometrically, whose isometry group is SOT$-$contractible according to Theorem 1.1.

Proposition 2.4. *Let (Ω, Σ, μ) be a positive measure space and we will regard $L^\infty(\mu)$ over the real field. Then $\mathcal{G}_{L^\infty(\mu)}$ is totally separated in the strong operator topology.*

Proof. For given $T, S \in \mathcal{G}_{L^\infty(\mu)}$ and $A \in \Sigma$ it holds that $T(\chi_A) \neq S(\chi_A)$ if and only if $||T\chi_A - S\chi_A|| \geq 1$ (see e.g. [10, Thm. 2]). Pick $T, S \in \mathcal{G}_{L^\infty(\mu)}$, $T \neq S$, if such exist. It is easy to find a set $C \in \Sigma$ such that $T\chi_C \neq S\chi_C$. Now,

$$\{U \in \mathcal{G}_{L^\infty(\mu)} :\ ||U\chi_C - T\chi_C|| < \frac{1}{2}\} \cup \{V \in \mathcal{G}_{L^\infty(\mu)} :\ ||U\chi_C - T\chi_C|| > \frac{1}{2}\} = \mathcal{G}_{L^\infty(\mu)}$$

is SOT$-$separation completing the claim. $\square$

To conclude, let us make a few remarks about the homotopy h appearing in the proof of Theorem 1.1. We note that a resembling transformation was applied in [3, p.251] in a different setting. The proof of Theorem 1.1 can easily be modified so that h becomes a homotopy on the set of linear isometric embeddings $L^p(\mu, \mathrm{X}) \to L^p(\mu, \mathrm{X})$, and hence this set is SOT-contractible for atomless μ and $p < \infty$. With equally small modifications it can be verified that the conclusion of Theorem 1.1 remains valid, if one investigates the contractibility of $\mathrm{GL}_{L^p(\mu,\mathrm{X})}$ in place of $\mathcal{G}_{L^p(\mu,\mathrm{X})}$.

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University of Helsinki, Department of Mathematics and Statistics, Box 68, (Gustaf Hällströminkatu 2b) FI-00014 University of Helsinki, Finland

E-mail address: talponen@cc.helsinki.fi